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GROUPS OF MEASURE-PRESERVING HOMEOMORPHISMS AND VOLUME-PRESERVING DIFFEOMORPHISMS OF NONCOMPACT MANIFOLDS AND MASS FLOW TOWARD ENDS

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1. Spaces of measures and groups of measure-preserving homeomorphisms

Suppose M is a connected n-manifold possibly with boundary. The symbol $\mathcal{B}(M)$ denotes the σ-algebra of Borel subsets of M.

Definition 1.1. A Radon measure on M is a Borel measure μ on M such that $\mu(K) < \infty$ for any compact subset K of M. A Radon measure μ is said to be good if

(i) $\mu(p) = 0$ for any point p of M and
(ii) $\mu(U) > 0$ for any nonempty open subset U of M.

Definition 1.2.

(1) $\mathcal{M}_g^{\partial}(M)$ denotes the set of good Radon measures on M wth $\mu(\partial M) = 0$.
(2) The weak topology w on $\mathcal{M}_g^{\partial}(M)$ is the weakest topology such that the function
$$\Phi_f : \mathcal{M}_g^{\partial}(M) \to \mathbb{R} : \ \Phi_f(\mu) = \int_M f\, d\mu$$
is continuous for any continuous function $f : M \to \mathbb{R}$ with compact support.

Let $\mathcal{H}(M)$ denote the group of homeomorphisms of M with the compact-open topology. Any subgroup $\mathcal{G}$ of $\mathcal{H}(M)$ is equipped with the subspace topology. $\mathcal{G}_0$ and $\mathcal{G}_1$ denote the connected component and the path-component of the identity in $\mathcal{G}$.

Definition 1.3. Suppose μ is a good Radon measures on M. The subgroups $\mathcal{H}(M;\mu) \subset \mathcal{H}(M;\mu\text{-reg}) \subset \mathcal{H}(M)$ are defined as follows:

(1) $h \in \mathcal{H}(M)$ is μ-preserving if $\mu(h(B)) = \mu(B)$ for any $B \in \mathcal{B}(M)$.
$\mathcal{H}(M;\mu)$ denotes the subgroup of $\mathcal{H}(M)$ consisting of μ-preserving homeomorphisms of M.

(2) $h \in \mathcal{H}(M)$ is μ-biregular if "$\mu(h(B)) = 0$ iff $\mu(B) = 0$ for any $B \in \mathcal{B}(M)$".
$\mathcal{H}(M; \mu\text{-reg})$ denotes the subgroup of $\mathcal{H}(M)$ consisting of μ-biregular homeomorphisms of M.

The topological group $\mathcal{H}(M)$ acts continuously on the space $\mathcal{M}_g^\partial(M)_w$ by $h \cdot \mu = h_*\mu$, where $h_*\mu \in \mathcal{M}_g^\partial(M)$ is defined by $(h_*\mu)(B) = \mu(h^{-1}(B))$ $(B \in \mathcal{B}(M))$. The subgroup $\mathcal{H}(M; \mu)$ coincides with the stabilizer of μ under this action.

We also use the following terminologies.

Definition 1.4. Suppose X is a space and A is a subspace of X.

(1) A is a SDR (strong deformation retract) of X if there exists a homotopy $\varphi_t : X \to X$ such that $\varphi_0 = id_X$, $\varphi_1(X) = A$ and $\varphi_t|_A = id_A$ $(0 \le t \le 1)$.
(2) A is HD (homotopy dense) in X if there exists a homotopy $\varphi_t : X \to X$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ $(0 < t \le 1)$.

In both cases the inclusion map $A \subset X$ is a homotopy equivalence with a homotopy inverse $\varphi_1 : X \to A$.

2. Compact case — Fathi's results

Suppose M is a compact connected n-manifold. The von Neumann-Oxtoby-Ulam theorem [10] asserts that the above action is essentially transitive.

Theorem 2.1. (von Neumann-Oxtoby-Ulam) *Suppose M is compact and μ, $\nu \in \mathcal{M}_g^\partial(M)$ with $\nu(M) = \mu(M)$. Then there exists $h \in \mathcal{H}_\partial(M)_0$ such that $h_*\mu = \nu$.*

A parametrized version of this theorem was obtained by A. Fathi [6]. Let $\mu \in \mathcal{M}_g^\partial(M)$. We need to restrict ourselves to the following subspace of $\mathcal{M}_g^\partial(M)$.

Definition 2.1. $\mathcal{M}_g^\partial(M; \mu\text{-reg})$ denotes the subset of $\mathcal{M}_g^\partial(M)$ consisting of $\nu \in \mathcal{M}_g^\partial(M)$ which has the same total mass and the same null sets as μ.

The action of $\mathcal{H}(M)$ on $\mathcal{M}_g^\partial(M)$ restricts to the action of the subgroup $\mathcal{H}(M;\ \mu\text{-reg})$ on the subspace $\mathcal{M}_g^\partial(M; \mu\text{-reg})_w$. We obtain the orbit map

$$\pi : \mathcal{H}(M;\ \mu\text{-reg}) \longrightarrow \mathcal{M}_g^\partial(M; \mu\text{-reg})_w \quad : \quad \pi(h) = h_*\mu.$$

Theorem 2.2. (A. Fathi [6], 1980) *Suppose M is a compact connected n-manifold.*

(1) *The orbit map π admits a section $\sigma : \mathcal{M}_g^\partial(M;\mu\text{-}reg)_w \to \mathcal{H}_\partial(M;\mu\text{-}reg)_1 \subset \mathcal{H}(M;\ \mu\text{-}reg)$.*

(2) $\mathcal{H}(M;\mu\text{-}reg) \cong \mathcal{H}(M;\mu) \times \mathcal{M}_g^\partial(M;\mu\text{-}reg)_w$

(3)
$$\mathcal{H}(M;\mu) \overset{SDR}{\subset} \mathcal{H}(M,\mu\text{-}reg) \overset{\text{Weak } HD}{\subset} \mathcal{H}(M)$$

(4) $n = 2$
$$\overbrace{\underset{ANR}{\mathcal{H}(M;\mu)} \overset{SDR}{\subset} \underset{ANR}{\mathcal{H}(M,\mu\text{-}reg)} \overset{HD}{\subset} \underset{ANR}{\mathcal{H}(M)}}^{SDR}$$

Corollary 2.1. (Yagasaki [13]) *$\mathcal{H}(M;\mu)$ is an ℓ_2-manifold.*

Corollary 2.1 easily follows from the next topological characterization of ℓ_2-manifold.

Theorem 2.3. (T. Dobrowolski - H. Toruńczyk [5])
A topological group G is a ℓ_2-manifold iff G is a separable, non locally compact, completely metrizable ANR.

3. Non-Compact Case — R. Berlanga's results

Suppose M is a noncompact connected n-manifold possibly with boundary. First we introduce some notations on the ends of M.

Definition 3.1.

(1) An end e of M is a function which assigns to each compact subset K of M a connected component $e(K)$ of $M - K$ such that $e(K_1) \supset e(K_2)$ if $K_1 \subset K_2$.

(2) $E(M)$ denotes the space of ends of M.
$\overline{M} = M \cup E(M)$ denotes the end compactification of M.

(3) The topology of $\overline{M}$ is described by the following conditions:
   (i) M is an open subspace of $\overline{M}$.
   (ii) Fundamental open neighborhoods of $e \in E(M)$ is given by
   $$N(e,K) = e(K) \cup \{e' \in E(M) \mid e'(K) = e(K)\} \qquad (K \subset M : \text{compact})$$

$\overline{M}$ is a compact metrizable space and $E(M)$ is a 0-dim compact subset of $\overline{M}$.
Let $\mu \in \mathcal{M}_g^\partial(M)$.

Definition 3.2.

(1) $e \in E(M)$ is μ-finite if $\mu(e(K)) < \infty$ for some compact subset K of M (i.e., e has a neighborhood with finite μ-mass).

(2) $E_f(M;\mu)$ denotes the subspace of μ-finite ends of M.

The von Neumann-Oxtoby-Ulam theorem is extended to the non-compact case in the following form.

Theorem 3.1. (R. Berlanga [1], 1983)

Suppose μ, $\nu \in \mathcal{M}_g^\partial(M)$ has same total mass and same finite ends. Then there exists $h \in \mathcal{H}_\partial(M)_1$ with $h_*\mu = \nu$.

A parametrized version of this theorem is obtained recently by R. Berlanga [3]. Simple examples show that the weak topology w on $\mathcal{M}_g^\partial(M;\mu\text{-reg})$ is not enough to extend the section theorem (Theorem 2.2 (1)) to the noncompact case. R. Berlanga introduces a little stronger topology called the finite-end weak topology, which turns out to be the correct topology for this purpose.

Definition 3.3. (Finite-end weak topology) Let $\mu \in \mathcal{M}_g^\partial(M)$.

(1) $\mathcal{M}_g^\partial(M;\mu\text{-end-reg})$ denotes the subset of $\nu \in \mathcal{M}_g^\partial(M)$ which has the same total mass, same null sets and same finite ends as μ.

(2) Consider the inclusions $M \overset{\iota}{\subset} M \cup E_f(M;\mu) \subset \overline{M}$.
The map ι induces the natural map

$$\iota_* : \mathcal{M}_g^\partial(M;\mu\text{-end-reg}) \longrightarrow \mathcal{M}_g^\partial(M \cup E_f(M;\mu))_w \ : \ \nu \longmapsto \overline{\nu} = \iota_*\nu$$

(3) The finite-end weak topology ew on $\mathcal{M}_g^\partial(M;\mu\text{-end-reg})$ is the weakest topology such that ι_* is continuous.

The space $\mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$ admits the contraction $\varphi_t(\nu) = (1-t)\nu + t\mu \quad (0 \le t \le 1)$.

Definition 3.4. $\mathcal{H}(M;\mu\text{-end-reg})$ denotes the subgroup of $\mathcal{H}(M)$ consisting of $h \in \mathcal{H}(M)$ which preserves μ-null sets and μ-finite ends of M.

The group $\mathcal{H}(M;\mu\text{-end-reg})$ acts continuously on $\mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew}$ by $h \cdot \nu = h_*\nu$ and we obtain the orbit map

$$\pi : \mathcal{H}(M;\mu\text{-end-reg}) \longrightarrow \mathcal{M}_g^\partial(M;\mu\text{-end-reg})_{ew} \ : \ \pi(h) = h_*\mu.$$

Theorem 3.2. (R. Berlanga [3], 2003)

(1) *The orbit map π has a section*

$$\sigma : \mathcal{M}_g^\partial(M;\mu\text{-}end\text{-}reg)_{ew} \longrightarrow \mathcal{H}_\partial(M;\mu\text{-}end\text{-}reg)_1 \subset \mathcal{H}(M;\mu\text{-}end\text{-}reg).$$

(2) $\mathcal{H}(M;\mu\text{-}end\text{-}reg) \cong \mathcal{H}(M;\mu) \times \mathcal{M}_g^\partial(M;\mu\text{-}end\text{-}reg)_{ew}$

(3) $$\mathcal{H}(M;\mu) \overset{SDR}{\subset} \mathcal{H}(M,\mu\text{-}end\text{-}reg) \subset \mathcal{H}(M)$$

The relation between the two groups $\mathcal{H}(M,\mu\text{-end-reg}) \subset \mathcal{H}(M)$ is not known for $n \geq 3$. In $n = 2$ we can apply our results on homeomorphism groups of noncompact 2-manifolds [11, 12] to obtain the following conclusions.

Theorem 3.3. (Yagasaki [13])

$$n = 2 \qquad \overbrace{\mathcal{H}(M;\mu)_0 \subset \mathcal{H}(M,\mu\text{-}end\text{-}reg)_0 \overset{\mathrm{HD}}{\subset} \mathcal{H}(M)_0}^{\mathrm{SDR}}$$

$$\begin{array}{ccc} \ell_2\text{-MFD} & \qquad\text{ANR}\qquad & \text{ANR} \end{array}$$

The main statement $\mathcal{H}(M,\mu\text{-end-reg})_0 \overset{\mathrm{HD}}{\subset} \mathcal{H}(M)_0$ can be derived by the following arguments. When M is a PL n-manifold, $\mathcal{H}^{\mathrm{PL}}(M)$ denotes the subgroup of $\mathcal{H}(M)$ consisting of PL-homeomorphisms of M.

(1) Suppose M is a noncompact connected 2-manifold. Then
  (i) M admits a PL-structure.
  (ii) $\mathcal{H}^{\mathrm{PL}}(M)_0$ is HD in $\mathcal{H}(M)_0$ for any PL-structure on M [12], cf. [7].

(2) Suppose M is a PL n-manifold and $\mu \in \mathcal{M}_g^\partial(M)$. Then the PL-structure on M can be isotoped to a new PL-structure so that $\mathcal{H}^{\mathrm{PL}}(M) \subset \mathcal{H}(M;\mu\text{-reg})$ [15].

4. Mass flow toward ends on non-compact n-manifolds

Suppose M is a noncompact connected n-manifold and $\mu \in \mathcal{M}_g^\partial(M)$.

4.1. **Topological Vector Space** $V_\mu(M)$.

First we define a topological vector space $V_\mu(M)$, which parametrizies mass flows toward ends by μ-preserving homeomorphisms.

Definition 4.1.

(1) $\mathcal{B}_c(M) = \{B \in \mathcal{B}(M) \mid \mathrm{Fr}\, B : \text{Compact}\}$

(2) $W(M)$ denotes the space of all functions $a : \mathcal{B}_c(M) \to \mathbb{R}$.

(i) $W(M)$ is a real vector space under the addition and the scalar product of real valued functions.

(ii) $W(M)$ is equipped with the product topology,

i.e., the topology induced by the projections

$$\pi_C : W(M) \to \mathbb{R} \ : \ \pi_C(a) = a(C) \qquad (C \in \mathcal{B}_c(M)).$$

(3) $V(M) = \{a : \mathcal{B}_c(M) \to \mathbb{R} \mid (*)_1, (*)_2, (*)_3\}$

$(*)_1$ $C, D \in \mathcal{B}_c(M),\ \ Cl(C - D), Cl(D - C) : \text{compact} \implies a(C) = a(D)$

$(*)_2$ $C, D \in \mathcal{B}_c(M),\ \ C \cap D = \emptyset \implies a(C \cup D) = a(C) + a(D)$

$(*)_3$ $a(M) = 0$

$V_\mu(M) = \{a \in V(M) \mid (*)_4\}$

$(*)_4$ $C \in \mathcal{B}_c(M),\ \ \mu(C) < \infty \implies a(C) = 0$

$V(M)$ and $V_\mu(M)$ are linear subspaces of $W(M)$, which are equipped with the subspace topology.

4.2. **Mass flow homomorphism toward ends** $J : \mathcal{H}_E(M, \mu) \to V_\mu(M)$.

Next we define a continuous group homomorphism $J : \mathcal{H}_E(M, \mu) \to V_\mu(M)$, which measures a mass moved toward ends by each $h \in \mathcal{H}_E(M, \mu)$. Let $E = E(M)$. Each $h \in \mathcal{H}(M)$ has a unique extension $\overline{h} \in \mathcal{H}(\overline{M})$.

Definition 4.2.

(1) $\mathcal{H}_E(M, \mu) = \{h \in \mathcal{H}(M, \mu) \mid \overline{h}|_E = id_E\}$ (a subgroup of $\mathcal{H}(M, \mu)$)

(2) $J : \mathcal{H}_E(M, \mu) \ni h \longmapsto J_h \in V_\mu(M)$

$$J_h(C) = \mu(C - h(C)) - \mu(h(C) - C) \quad (C \in \mathcal{B}_c(M))$$

The group $\mathcal{H}_E(M, \mu)$ acts continously on $V_\mu(M)$ by $h \cdot a = J_h + a$ and the homomorphism $J : \mathcal{H}_E(M, \mu) \to V_\mu(M)$ coincides with the orbit map at $0 \in V_\mu(M)$.

Theorem 4.1. (Yagasaki [14])

(1) *The map J admits a section $s : V_\mu(M) \to \mathcal{H}_\partial(M,\mu)_1 \subset \mathcal{H}_E(M,\mu)$* (i.e., $Js = id$) *with $s(0) = id_M$.*

(2) (i) $\mathcal{H}_E(M;\mu) \cong \operatorname{Ker} J \times V_\mu(M)$ (ii) $\operatorname{Ker} J \subset \mathcal{H}_E(M;\mu)$ *: a SDR*

$\operatorname{Ker} J$ contains the subgroup $\mathcal{H}^c(M;\mu)$ of μ-preserving homeomorphisms with compact support. Our next aim is the study of relation between these groups.

5. Spaces of volume forms and groups of volume-preserving diffeomorphisms

Suppose M is a connected oriented C^∞ n-manifold without boundary.

Definition 5.1.

(1) $\mathcal{D}^+(M)$ deotes the group of orientation-preserving diffeomorphisms of M with the compact-open C^∞-topology.

(2) For a positive volume form ω on M,

$\mathcal{D}(M;\omega)$ denotes the subgroup of ω-preserving diffeomorphisms of M.

(3) $\mathcal{V}^+(M)_w$ denotes the space of positive volume forms on M equipped with the weak C^∞ topology.

For $m \in (0,\infty]$, $\mathcal{V}^+(M,m)_w = \{\mu \in \mathcal{V}^+(M) \mid \mu(M) = m\}$ (the weak C^∞ topology).

Each $\mu \in \mathcal{V}^+(M)$ determines a unique good Radon measure on M, which is denoted by the same symbol μ. This defines an inclusion $\mathcal{V}^+(M) \subset \mathcal{M}_g^\partial(M)$.

The topological group $\mathcal{D}^+(M)$ acts continuously on $\mathcal{V}^+(M)_w$ and $\mathcal{V}^+(M,m)_w$ by $h \cdot \mu = h_*\mu$ $(= (h^{-1})^*\mu)$. The subgroup $\mathcal{D}(M;\omega)$ coincides with the stabilizer of ω under this action.

5.1. **Compact case.**

Suppose M is a compact connected oriented C^∞ n-manifold without boundary. Moser's theorem [9] implies the transitivity of this action and its parametrized version.

Theorem 5.1. *Suppose M is a compact connected oriented C^∞ n-manifold.*

(1) (Transitivity) *For any μ, $\nu \in \mathcal{V}^+(M,m)$ there exists $h \in \mathcal{D}(M)_1$ such that $h_*\mu = \nu$.*

(2) (Parametrized version) *Let $\omega \in \mathcal{V}^+(M;m)$. Then the orbit map $\pi : \mathcal{D}^+(M) \to \mathcal{V}^+(M;m)_w$, $\pi(h) = h_*\omega$, admits a section $\sigma : \mathcal{V}^+(M;m)_w \longrightarrow \mathcal{D}(M)_1 \subset \mathcal{D}^+(M)$.*

5.2. **Non-compact case.**

Suppose M is a non-compact connected C^∞ n-manifold without boundary. Recall that $E = E(M)$ is the space of ends of M and $\overline{M} = M \cup E(M)$ is the end compactification of M. Each $h \in \mathcal{D}(M)$ has a unique extension $\overline{h} \in \mathcal{H}(\overline{M})$. For $\mu \in \mathcal{V}^+(M)$, $E_f(M,\mu)$ denotes the subspace of $E(M)$ consisting of μ-finite ends of M.

Definition 5.2. Suppose $F \subset E(M)$ is an open subset.

(1) $\mathcal{D}^+(M;F) = \{h \in \mathcal{D}^+(M) \mid \overline{h}(F) = F\}$ (a subgroup of $\mathcal{D}^+(M)$)

(2) $\mathcal{V}^+(M;F) = \{\mu \in \mathcal{V}^+(M) \mid E_f(M,\mu) = F\}$

$\mathcal{V}^+(M;m,F) = \mathcal{V}^+(M;m) \cap \mathcal{V}^+(M;F)$

$\mathcal{M}_g^\partial(M;F) = \{\mu \in \mathcal{M}_g^\partial(M) \mid E_f(M,\mu) = F\}$

(3) (Finite-end weak topology)

The inclusion $M \overset{\iota}{\subset} M \cup F$ $(\subset \overline{M})$ induces the injection

$$\begin{array}{ccccccc} \iota_\# & : & \mathcal{V}^+(M;m,F) & \subset & \mathcal{M}_g^\partial(M;F) & \xrightarrow{\iota_*} & \mathcal{M}_g(M \cup F)_w \, . \\ & & \nu & \longmapsto & \nu & \longmapsto & \overline{\nu} = \iota_*\nu \end{array}$$

The finite-end weak topology ew on $\mathcal{V}^+(M;m,F)$ is the weakest topology such that the maps $\iota_\#$ and $id : \mathcal{V}^+(M;m,F) \to \mathcal{V}^+(M;m,F)_w$ are continuous.

The group $\mathcal{D}^+(M;F)$ acts continuously on $\mathcal{V}^+(M;m,F)_{ew}$ by $h \cdot \mu = h_*\mu$ and the stabilizer of $\omega \in \mathcal{V}^+(M;m,F)_w$ coincides with the subgroup $\mathcal{D}(M;\omega)$. Transitivity of this action was verified by R. E. Greene - K. Shiohama [8].

Theorem 5.2. (R. E. Greene - K. Shiohama [8])

For any μ, $\nu \in \mathcal{V}^+(M;m,F)$ there exists $h \in \mathcal{D}(M)_1$ such that $h_*\mu = \nu$.

A C^∞-modification of R. Berlanga's argument [3] leads to the parametrized version of this theorem.

Theorem 5.3. (Yagasaki [15])
Suppose P is a paracompact Hausdorff space and $\mu, \nu : P \to \mathcal{V}^+(M; F)_{ew}$ are maps such that $\mu_p(M) = \nu_p(M)$ $(p \in P)$. Then there exists a map $h : P \to \mathcal{D}(M)_1$ such that

(i) $h_{p*}\mu_p = \nu_p$ $(p \in P)$ *and* (ii) *if $p \in P$ and $\mu_p = \nu_p$, then $h_p = id_M$.*

Corollary 5.1. *Let $\omega \in \mathcal{V}^+(M; m, F)$.*

(1) *The orbit map $\pi : \mathcal{D}^+(M; F) \longrightarrow \mathcal{V}^+(M; m, F)_{ew}$, $\pi(h) = h_*\omega$, admits a section $\sigma : \mathcal{V}^+(M; m, F)_{ew} \to \mathcal{D}(M)_1 \subset \mathcal{D}^+(M; F)$.*

(2) (i) $\mathcal{D}^+(M; F) \cong \mathcal{V}^+(M; m, F)_{ew} \times \mathcal{D}(M; \omega)$ (ii) $\mathcal{D}(M; \omega) \subset \mathcal{D}^+(M; F)$ *: a SDR*

5.3. **Mass flow toward ends on non-compact C^∞ n-manifolds.**

Suppose M is a non-compact connected C^∞ n-manifold without boundary and $\omega \in \mathcal{V}^+(M)$. The topological vector space $V(M)$, $V_\omega(M)$ and a continuous group homomorphism $J^\omega : \mathcal{D}_E(M, \omega) \to V_\omega(M)$ are defined as in § 4.1 and § 4.2. For $h \in \mathcal{D}_E(M; \omega)$

$$J_h^\omega : \mathcal{B}_c(M) \to \mathbb{R} : \quad J_h^\omega(C) = \omega(C - h(C)) - \omega(h(C) - C) \quad (C \in \mathcal{B}_c(M)).$$

The group $\mathcal{D}_E(M, \omega)$ acts continuously on $V_\omega(M)$ by

$$h \cdot a = J_h^\omega + a \qquad (h \in \mathcal{D}_E(M, \omega),\ a \in V_\omega(M)).$$

The map $J^\omega : \mathcal{D}_E(M, \omega) \to V_\omega(M)$ coincides with the orbit map at $0 \in V_\omega(M)$.

Definition 5.3. For two maps $\mu, \nu : P \to \mathcal{V}^+(M)$ we write as $\mu \underset{c}{\sim} \nu$ if

for any $p \in P$ there exists a neighborhood U of p in P and a compact subset $K \subset M$ such that $\mu_q = \nu_q$ on $M - K$ for any $q \in U$.

Theorem 5.4. *Suppose P is a paracompact Hausdorff space and $\mu, \nu : P \to \mathcal{V}^+(M)_w$, $a : P \to V(M)$ are maps such that $\mu \underset{c}{\sim} \nu$, $(\mu - \nu)(M) = 0$ and $a_p \in V_{\mu_p}(M)$ $(p \in P)$. Then there exists a map $h : P \to \mathcal{D}(M)_1$ such that*

(i) $h_{p*}\mu_p = \nu_p$ $(p \in P)$ *and* (ii) *if $p \in P$ and $\mu_p = \nu_p$, then $J_{h_p}^{\mu_p} = a_p$.*

Corollary 5.2. *Let $\omega \in \mathcal{V}^+(M)$.*

(1) *The map $J^\omega : \mathcal{D}_E(M, \omega) \to V_\omega(M)$ admits a section $s : V_\omega(M) \to \mathcal{D}(M, \omega)_1 \subset \mathcal{D}_E(M, \omega)$ ($J^\omega s = id_{V_\omega(M)}$) with $s(0) = id_M$.*

(2) (i) $\mathcal{D}_E(M; \omega) \cong \operatorname{Ker} J^\omega \times V_\omega(M)$ (ii) $\operatorname{Ker} J^\omega \subset \mathcal{D}_E(M; \omega)$ *: a SDR*

Our next aim is to study the relation between two groups $\mathcal{D}^c(M; \omega) \subset \operatorname{Ker} J^\omega$.

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